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Fermion families and chirality through extra dimensions

R. Erdem[a]

Department of Physics, Izmir Institute of Technology, Gülbahçe Köyü, Urla, 35437 Izmir, Turkey

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Abstract. We give a simple model to explain the origin of fermion families and chirality through the use of a domain wall–anti-domain wall pair placed in a five dimensional space-time.

Higher dimensional spaces have been revived in the last years by the hope that the extra space dimensions other than the usual three dimensions may be experimentally accessible in the near future. In this context the idea of the confinement of the usual particle spectrum into a four dimensional topological defect of the higher dimensional space-time is also revived especially in the view that by using Randall–Sundrum [1] type spaces one can confine gravity as well in infinite dimensions. In this study we shall consider a metric with a four dimensional Poincaré invariance [2] and a domain wall structure in a five dimensional space. We find that it gives some important clues towards the understanding of the origin of fermion families and chirality. By using a Randall–Sundrum-like metric we get a domain wall and an anti-domain wall solution in five dimensions. The graviton modes are localized in a narrow width in the five dimensions so that the resulting space-time at low energies is the usual four dimensional space-time. The classical background scalar fields associated with the domain wall and the anti-domain wall couple to fermions so that their interaction with fermions is effectively like a single domain wall. The metric can be written in such a way that there is more than one family. After adding a mass-like term to this scheme the left handed and right handed fermions become concentrated at different regions in the wall. In other words, the wall itself acts as a mother 3-brane which carries two sub-branes, the right handed and the left handed ones. Each of these two sub-branes contains n different symmetrical sub-sub-branes whose locations can be identified with different fermion families. The framework employed here has some conceptual similarities with the study of Dvali and Shifman, which considers families as neighbors in a multi-brane world in five dimensions [3]. Another study with some similar aspects is by Arkani-Hamed and Schmaltz [4] where fermion mass hierarchies are explained by the exponentially suppressed overlap of fermion wave functions located at different points in the extra dimension(s). We will give a comparison of these studies with the present one at the end of this study.

Consider the following Lagrangian:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}g^{BC}\partial_B\phi_1\partial_C\phi_1 + \frac{1}{2}g^{BC}\partial_B\phi_2\partial_C\phi_2 \\ &+ \frac{3\mu^2}{2\lambda}\mathrm{e}^{2\sqrt{\lambda}/\mu\phi_1} + \frac{3\mu^2}{2\lambda}\mathrm{e}^{-2\sqrt{\lambda}/\mu\phi_2} \\ &+ V_1(\sigma)(|\phi_1| - \phi_3) + V_2(\sigma)(|\phi_2| - \phi_3) \end{aligned} \qquad (1)$$

with

$$\begin{aligned}\mathrm{d}s^2 &= g_{AB}\mathrm{d}x^A\mathrm{d}x^B \\ &= \mathrm{e}^{2A}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - (3ay^2+b)^2\mathrm{e}^{2B}\mathrm{d}y^2, \qquad (2)\\ \text{where } A &= -\tanh\eta, \quad B = -2\ln\mathrm{Cosh}\eta - \tanh\eta, \\ \delta_{\mu\nu} &= \mathrm{diag}(1,-1,-1,-1), \quad \mu = 0,1,2,3,\end{aligned}$$

and where $\eta = \eta(y)$ is a function of y, $V(\sigma)$ is a potential depending on the auxiliary field σ; a, b are some constants. The fields σ and ϕ_3 are auxiliary fields with no kinetic terms and no contribution to the energy-momentum tensor and no interaction terms with physical fields. Their only role here is to make sure that the equations of motion for $\phi_{1(2)}$ give domain wall and anti-domain wall solutions i.e. $\tanh\eta$ and $-\tanh\eta$, respectively, as we shall see. The $V_{1(2)}(\sigma)$ act like external source terms which help the stabilization of the resulting five dimensional space-time and they can be taken as external source terms which result from an effective potential induced by these fields and/or some additional fields. Although ϕ_1 by itself is enough to localize the fermions the presence of ϕ_2 simplifies the energy-momentum tensor, and hence the form of the metric. We shall see that the solutions of the equations of motion in the presence of the metric given in (2) imposes $\eta = ay^3 + by + c$ (where c is some constant). Hence the localization of the fermion wave function profile at $\eta = \eta_0$ corresponds to three identical wave function profiles localized at $y = y_1$, $y = y_2$, $y = y_3$. These three identical wave function profiles are identified as three generations of a fermion family. All these issues will be discussed in detail in the following paragraphs. Another remark is

[a] e-mail: erdem@likya.iyte.edu.tr

that a simple-minded consideration may suggest that one may redefine the fifth direction so that the coefficients in front of $\mathrm{d}y^2$ may be absorbed into the redefinition of the fifth coordinate, but this is not true in general. Such a reparametrization needs a change in the form of the energy-momentum tensor in the Einstein field equations as well. For example, if we replace the coefficient of the $\tanh\eta$ term of B in (2) by 2 the metric sets the four dimensional part of the energy-momentum tensor to zero, while a reparametrization $-(3ay^2+b)^2\mathrm{e}^{2B}\mathrm{d}y^2 \to \mathrm{d}u^2$ leads to a non-zero four dimensional energy-momentum tensor. If we had chosen a metric where a rescaling of the y does not change the physical outcome then the metric here would be equivalent to the usual Randall–Sundrum model. So it is this non-trivial form of the metric which accounts for additional phenomenology such as fermion generations and chirality. After these remarks let us continue to study the model. The equations of motion corresponding to the solutions which depend only on the y-direction are

$$-\frac{\partial}{\partial y}\left[\left(\frac{1}{(3ay^2+b)^2}\right)(1-\tanh^2\eta)^{-2}\mathrm{e}^{2\tanh\eta}\frac{\partial}{\partial y}\phi_{1(2)}\right]$$
$$-(+)3\frac{\mu}{\sqrt{\lambda}}\mathrm{e}^{(2\sqrt{\lambda}/\mu)[+(-)\phi_{1(2)}]} = V_{1(2)}(\sigma),$$
$$\frac{\partial V_1(\sigma)}{\partial\sigma}(|\phi_1|-\phi_3)+\frac{\partial V_2(\sigma)}{\partial\sigma}(|\phi_2|-\phi_3)=0. \quad (3)$$

These equations have domain wall and anti-domain wall solutions [5,6] given by

$$\phi_1=\phi_3=\phi_{cl}=\frac{\mu}{\sqrt{\lambda}}\tanh\eta,\quad \phi_2=\phi_{\mathrm{Acl}}=-\frac{\mu}{\sqrt{\lambda}}\tanh\eta$$
$$\eta=(ay^3+by+c) \quad (4)$$

provided $V_1(\sigma)=-V_2(\sigma)$ and

$$V_1(\sigma)=\frac{\mu}{\sqrt{\lambda}} \quad (5)$$
$$\times\left[-5\mathrm{e}^{2\tanh\eta}+\frac{\eta''}{(\eta')^2(1-\tanh^2\eta)}-\frac{2\tanh\eta}{(1-\tanh^2\eta)}\right],$$

where the superscripts $'$ and $''$ denote the first and the second derivatives with respect to y, c is a constant of integration, and μ, λ are constants of dimension $[\mathrm{length}]^{-1}$, $[\mathrm{length}]^{1}$, respectively, so that ϕ has the correct dimension. At this point a remark is in order. The term $V_{1(2)}(\sigma)$ $(|\phi_{1(2)}|-\phi_3)$ in (1) is linear in $\phi_{1(2)}$ and for most of the values of η, $V_{1(2)}(\sigma)$ is negative (positive). So $V_1(\sigma)(|\phi_1|-\phi_3)$ is repulsive, which could destabilize the localization of the matter corresponding to ϕ_1. However the term $(3\mu^2)/(2\lambda)\mathrm{e}^{(2\sqrt{\lambda}/\mu)\phi_1}$ $((3\mu^2)/(2\lambda)\mathrm{e}^{-(2\sqrt{\lambda}/\mu)\phi_2})$ is repulsive (attractive) for all values of $\phi_{1(2)}$ provided $(3\mu^2)/(2\lambda)>0$. So the overall system is stable. In fact this is evident from the localization of the corresponding energy-momentum tensor in the fifth direction as we shall see when we study the Einstein equations below. This can be stated in terms of the quantum field theory language as follows. The tadpole-like terms in the potential tend to shift the vacuum state. However the very existence of the exponential terms tends to suppress such a shift of the vacuum state. Therefore the vacuum state is stable.

We assume that the vacuum has two background fields consisting of the domain wall and the anti-domain wall given in (4) where both are centered at $\eta=0$. Then effectively the classical Lagrangian in the energy-momentum tensor is

$$\mathcal{L}_{\mathrm{cl}}=\frac{1}{2}g^{BC}\partial_B\phi_{\mathrm{cl}}\partial_C\phi_{\mathrm{cl}}+\frac{1}{2}g^{BC}\partial_B\phi_{\mathrm{Acl}}\partial_C\phi_{\mathrm{Acl}}$$
$$+\frac{3\mu^2}{2\lambda}\mathrm{e}^{(2\sqrt{\lambda}/\mu)\phi_{\mathrm{cl}}}+\frac{3\mu^2}{2\lambda}\mathrm{e}^{(-2\sqrt{\lambda}/\mu)\phi_{\mathrm{Acl}}}+V(\sigma)(\phi_{\mathrm{cl}}+\phi_{\mathrm{Acl}})$$
$$=2\frac{\mu^2}{\lambda}\mathrm{e}^{2\tanh\eta}. \quad (6)$$

The action relevant to gravity is

$$S=\int \mathrm{d}^5x\sqrt{-G}(R+\Lambda+\mathcal{L}_{\mathrm{cl}}), \quad (7)$$

where $\mathcal{L}_{\mathrm{cl}}$ stands for the Lagrangian in terms of the classical fields ϕ_{cl} and ϕ_{Acl} is given in (5), G is the five dimensional metric tensor, and R is the five dimensional Ricci scalar, and Λ stands for the cosmological constant in the bulk. The corresponding Einstein equations are

$$R_{AB}-\frac{1}{2}G_{AB}R$$
$$=\frac{1}{4M^3}[G_{AB}(\Lambda+\mathcal{L}_{\mathrm{cl}})-\partial_A\phi_{\mathrm{cl}}\partial_B\phi_{\mathrm{cl}}-\partial_A\phi_{\mathrm{Acl}}\partial_B\phi_{\mathrm{Acl}}]$$
$$=\frac{1}{4M^3}\left[G_{AB}\left(\Lambda+2\frac{\mu^2}{\lambda}\mathrm{e}^{2\tanh\eta}\right)\right.$$
$$\left.-\partial_A\phi_{\mathrm{cl}}\partial_B\phi_{\mathrm{cl}}-\partial_A\phi_{\mathrm{Acl}}\partial_B\phi_{\mathrm{Acl}}\right], \quad (8)$$

where M is the five dimensional Planck mass. The Einstein equations [7,8] for the metric in (2) are satisfied for all y provided

$$\Lambda=0,\quad 2\lambda M^3=\mu^2. \quad (9)$$

The graviton zero-modes are confined to a region near the center of the domain wall–anti-domain wall pair. In order to see this we study the equation of motion for graviton zero-modes. As in [1] we write the metric tensor with linearized quantum fluctuations included as $g_{MN}=G_{MN}+h_{MN}$. h can be written as $h_{MN}=\epsilon_{MN}(y)\mathrm{e}^{\mathrm{i}p\cdot x}$ where $p^2=m_g^2$ stands for the mass of the graviton modes. We know that $h_{\mu\nu}$ must have a massless mode corresponding to the usual gravity. So this graviton zero-mode must be confined to the brane in order to prevent any conflict with the inverse square law of gravity. For this purpose one must write the linearized equation of motion for $h_{\mu\nu}$. We work in the gauge $\partial^\mu h_{\mu\nu}=h^\mu_\mu=0$ as in [1]. We expand the four dimensional metric tensor as $g_{\mu\nu}=\mathrm{e}^{-2\tanh\eta(y)}\delta_{\mu\nu}+h_{\mu\nu}$ and $g_{55}=-(3ay^2+b)^2(1-\tanh^2\eta)\mathrm{e}^{-2\tanh\eta(y)}+h_{55}$, where $\delta_{\mu\nu}$ is the Minkowski metric tensor. The equation of motion for $h_{\mu\nu}$ is

$$\left[-\frac{m_g^2}{2}\mathrm{e}^{2\tanh\eta(y)}\frac{1}{2(3ay^2+b)^2}(1-\tanh^2\eta)^{-2}\right.$$

$$\times\, \mathrm{e}^{2\tanh\eta(y)}\partial_y^2 - \frac{2\mu^4}{\lambda}\mathrm{e}^{2\tanh\eta}\Big]\,\epsilon_{\mu\nu} = 0. \quad (10)$$

This is equivalent to

$$\left[-\frac{1}{2}\partial_y^2 + V(y)\right]\epsilon_{\mu\nu} = 0, \quad (11)$$

where

$$V(y) = -\left(\frac{2\mu^2}{\lambda} + \frac{1}{2}m_g^2\right)(3ay^2+b)^2(1-\tanh^2\eta)^2. \quad (12)$$

Analytical calculations and the use of commercially available software (Mathematica) shows that this potential has three minima in general; one at $y=0$, the others at a positive and a negative y (the one at $y=0$ may disappear for some values of a,b,c). For the range of parameters where $|a| >> |b|,|c|$ (which corresponds to a sufficiently narrow brane) there are two deep minima on either side of $y=0$. For example for the parameters $a=-10$, $b=0.007$, $c=-0.45006$ there are only two minima: at $y\simeq-0.45$, $y\simeq 0.45$. The one at $y\simeq-0.45$ ($\eta\simeq-0.46$) is about six times deeper than the one at $y\simeq 0.45$. So both massless and massive graviton modes are effectively localized in the usual four dimensional space. We shall see in the following paragraphs that while the fermions are localized at $\eta\simeq 0$ it need not be exactly at the same region as the graviton zero-modes. In this way one can account for why the gravitational attraction is small in our universe while the graviton zero-mode is localized in the fifth dimension.

We take the following fermion-scalar interaction Lagragian:[1]

$$\begin{aligned}
&\mathrm{i}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \mathrm{i}\bar{\Psi}\Gamma^4\partial_4\Psi + +g_1\bar{\Psi}\phi_1\Psi + g_2\bar{\Psi}\phi_2\Psi\\
&\quad = \mathrm{i}\bar{\Psi}\gamma^\mu\mathrm{e}^{\tanh\eta}D_\mu\Psi\\
&\quad +\mathrm{i}\bar{\Psi}(-\mathrm{i}\gamma_5)\frac{1}{(3ay^2+b)}(1-\tanh^2\eta)^{-1}\mathrm{e}^{\tanh\eta}\frac{\partial\Psi}{\partial y}\\
&\quad +g_1\bar{\Psi}\phi_1\Psi + g_2\bar{\Psi}\phi_2\Psi, \qquad (13)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^\mu &= \mathrm{e}^{\tanh\eta}\gamma^\mu,\\
\Gamma^4 &= -\mathrm{i}\gamma_5\frac{\mathrm{e}^{\tanh\eta}}{(3ay^2+b)}(1-\tanh^2\eta)^{-1},\\
D_\mu &= \partial_\mu + \mathrm{i}\frac{1}{2}gB_\mu.
\end{aligned}$$

Hence in the presence of a background consisting of a domain wall–anti-domain wall pair, the five dimensional Dirac equation is

$$\begin{aligned}
&\mathrm{i}\mathrm{e}^{\tanh\eta}\gamma^\mu D_\mu\Psi + \mathrm{e}^{\tanh\eta}\frac{1}{3ay^2+b}(1-\tanh^2\eta)^{-1}\gamma_5\frac{\partial\Psi}{\partial y}\\
&\quad +g\mathrm{e}^{-\tanh\eta}\phi_{\mathrm{cl}}\Psi = 0, \qquad (14)
\end{aligned}$$

where $g=g_1-g_2$ and we have used $\phi_{\mathrm{Acl}} = -\phi_{\mathrm{cl}}$. We consider the solutions which propagate in the usual four dimensions as free fields whose form (for the metric in (2)) is

$$\Psi = \mathrm{e}^{-\mathrm{i}[(\mathrm{e}^{2A})(p_0x_0-\vec{p}\cdot\vec{x})]}\chi(y). \quad (15)$$

At $\eta=0$ (14) for the free field solution becomes

$$[\gamma^\mu p_\mu\Psi + m\gamma_5]\Psi = 0, \quad (16)$$

where

$$m = \left(\frac{\partial\chi}{\partial\eta}\right)\Big|_{\eta=0} = \begin{pmatrix} m_{\mathrm{L}} & 0\\ 0 & m_{\mathrm{R}}\end{pmatrix}, \quad \beta = g\frac{\mu}{\sqrt{\lambda}}.$$

In fact, the form of (16) is true in the neighborhood of any point η, as can be checked. Then after replacing (15) and (16) in (14) we get

$$\begin{aligned}
&-m\mathrm{e}^{-\tanh\eta}\gamma_5\Psi + \frac{1}{3ay^2+b}(1-\tanh^2\eta)^{-1}\mathrm{e}^{\tanh\eta}\gamma_5\frac{\partial\Psi}{\partial y}\\
&\quad +\beta\tanh\eta\Psi = 0, \qquad (17)
\end{aligned}$$

which may be written in terms of $\Psi_{\mathrm{L}} = (1/2)(1-\gamma_5)\Psi$, $\Psi_{\mathrm{R}} = (1/2)(1+\gamma_5)\Psi$ as

$$\begin{aligned}
&\frac{\partial\Psi_{\mathrm{L}}}{\partial y} - \frac{\partial\Psi_{\mathrm{R}}}{\partial y} + \eta'(1-\tanh^2\eta)\\
&\times\Big[(\beta\tanh\eta\mathrm{e}^{-\tanh\eta} - m_{\mathrm{L}}\mathrm{e}^{-2\tanh\eta})\Psi_{\mathrm{L}}\\
&+(\beta\tanh\eta\mathrm{e}^{-\tanh\eta} + m_{\mathrm{R}}\mathrm{e}^{-2\tanh\eta})\Psi_{\mathrm{R}}\Big] = 0. \qquad (18)
\end{aligned}$$

The solutions of (18) are

$$\begin{aligned}
\Psi_{\mathrm{R}} &= \exp\left[-\frac{1}{2}m_{\mathrm{R}}\mathrm{e}^{-2\tanh\eta} - \beta(1+\tanh\eta)\mathrm{e}^{-\tanh\eta}\right]\psi_{\mathrm{R}},\\
\Psi_{\mathrm{L}} &= \exp\left[-\frac{1}{2}m_{\mathrm{L}}\mathrm{e}^{-2\tanh\eta} + \beta(1+\tanh\eta)\mathrm{e}^{-\tanh\eta}\right]\psi_{\mathrm{L}},
\end{aligned} \quad (19)$$

where ψ is the solution of $(\mathrm{i}\gamma^\mu\partial_\mu + m\gamma_5 - \beta\tanh\eta_0)\psi = 0$. At this point we want to make two important remarks. The first remark is that the term with the coefficient m in (16) is not a mass term; it has a γ_5 in front. So it may lead to localized solutions for both Ψ_{L} and Ψ_{R} as in (19). In fact, due to this reason it may lead to localized solutions for both Ψ_{L} and Ψ_{R} even in the five dimensional Minkowski space. The second remark is that provided we use this metric even a mass term would not lead to a non-localized solution for either of Ψ_{L}, Ψ_{R} because a mass term results in a $\pm m, \mathrm{e}^{-2,\tanh\eta}$ term in the exponent in (19) and $\tanh\eta$ is finite for all values of η. After these remarks we return to the analysis of the solutions in (19). The extrema of (19) are at

$$\begin{aligned}
\tanh\eta\mathrm{e}^{\tanh\eta} &= \frac{m_{\mathrm{L}}}{\beta} \quad \text{for } \Psi_{\mathrm{L}},\\
\tanh\eta\mathrm{e}^{\tanh\eta} &= -\frac{m_{\mathrm{R}}}{\beta} \quad \text{for } \Psi_{\mathrm{R}}.
\end{aligned} \quad (20)$$

[1] A similar Lagrangian is considered in [9]. In fact one can identify ϕ in this equation as the gauge field corresponding to a sixth dimension. This can be, for example, done by embedding this five dimensional space in a six dimensional space as studied by Manton [10] where instead of taking the extra dimensions compact one should only assume rotational symmetry. In that case $B_5=\phi$ in (13) should be replaced by $\Phi+\tilde{\Phi}$ of [10]. However for the sake of simplicity we take this term to arise from a general scalar-fermion interaction term

The consistence of these two equations implies $|m/\beta| < \mathrm{e}^2$. The extremum of Ψ_{L} is a maximum while that of Ψ_{R} is a minimum for sufficiently small (m/β) (while they may be outside of the domain of the functions for some values of the parameters). The shape of the graphs for the functions in (20) obtained by our Mathematica software for several values of the parameters verifies this conclusion. The shape of the graphs of these functions is a narrow Gaussian-like curve for Ψ_{L} and a cup-like curve for Ψ_{R}. (Although $\Psi_{\mathrm{L(R)}}$ does not go to zero as $\eta \to \infty$, this does not pose a problem because the square root of the determinant of the metric tensor (i.e $\eta'(1-\tanh^2\eta)\mathrm{e}^{-\tanh\eta}$) in the integration measure of the normalization integration of the wave function can be used to transform the integration into an integration in terms of $u = \tanh\eta$ so that the result of the integration is finite.) We observe that the η dependent part of Ψ_{R} is highly suppressed with respect to Ψ_{L} for most of the values (if not for all the values of η). One gets phenomenologically interesting values for some of the values of the parameters. If one assumes that the photon is localized in a narrow range of η where the magnitude of the Ψ_{L} and Ψ_{R} are almost the same while the gauge bosons of the weak interactions can penetrate into the bulk more deeply (where the average magnitude of Ψ_{R} is suppressed with respect to that of Ψ_{L}) then one may explain why the electromagnetic interactions are vector-like, while weak interactions are chiral. For example for $m_{\mathrm{L}} = 3$, $m_{\mathrm{R}} = -0.3$ $\beta = 3$ the average magnitude of the y dependent part of Ψ_{R} is about the same as Ψ_{L} at $-0.4 < \eta < -0.3$, while for most of the values of η the y dependent part in Ψ_{L} is much greater than that of Ψ_{R}. One may assume that the photon is localized about $\eta \simeq -0.35$ (e.g. in the interval $-0.37 < \eta < -0.33$), while the weak bosons propagate in the region where $-0.3 < \eta < 0.4$. The average density of Ψ_{L} in the region $-0.3 < \eta < 0.4$ is much higher (about 30 times) than that of Ψ_{R}. This may explain why the right handed weak currents are highly suppressed with respect to the left handed ones. The fact that the neutral weak currents have a right handed component while the charged weak currents are purely left handed may be accounted for if we assume that the wave function of the W bosons (compared to the wave function of the Z boson) is localized in a smaller region, where the average value of Ψ_{L} is much greater than the average value of Ψ_{R} when compared to the broader region where the Z boson is localized. For example if we take $m_{\mathrm{L}} = 3$, $m_{\mathrm{R}} = -0.3$, $\beta = 3$ and assume that Z is localized in the region $-0.3 < \eta < 0.4$, while the W bosons are localized in $0 < \eta < 0.3$, then the average value of ψ_{L} interacting with Z bosons is about 15 times that of the average value of Ψ_{R} while the average value of Ψ_{L} interacting with W bosons is about 50 times that of the Ψ_{R}. So in this way Z bosons have an appreciable amount of vector interactions, while W bosons are effectively purely left handed.

Although we just simply assume that W bosons are better localized than Z bosons, one may also give plausibility arguments for this assumption. For example, one may assume that the profile of the wave functions for W^+ and W^- are a little bit separated in the fifth dimension. Then the Coulomb interaction between W^+ and W^- leads to the displacement of their wave functions towards each other to better localize the wave function profile in the same way as in [11], where the effect of the Coulombic interactions between quarks leads to a better localization of the wave function profiles in the fifth dimension. Of course one needs a separate study to see if this effect can produce a sufficient localization for the W bosons. In any case this example shows that the assumption of the better localization of W bosons with respect to Z bosons is a plausible assumption and we shall assume here that this is really the case. At this step I want to make a remark. The η parameter is dimensionless and the magnitude of the dimensionful parameter y corresponding to some value of $\eta = ay^3 + by + c$ depends on the parameters a, b, c. For example, if one takes $|a| >> |b|, |c|$ then the true width of the brane corresponding to $-0.5 < \eta < 0.5$ is of the order of $1/a^{1/3}$ which may be extremely small if we take a very large.

In fact, each of the curves describing the η dependence of Ψ_{L} and Ψ_{R} corresponds to three curves, the same in form but translated in the y-direction because to each value of $\eta = ay^3 + by + c$ there correspond three values of y in general. We assume that each of these equations $\eta(y) = z_{\mathrm{L(R)}}$ (where $z_{\mathrm{L(R)}}$ denotes the values of η overlapping with our brane) has three distinct real roots for each value of $z_{\mathrm{L(R)}}$. As long as we choose the width of the brane in the fifth dimension sufficiently small we can find such pieces of curves provided the equation $\eta = z_{\mathrm{L(R)}}$ has three distinct real roots for one value of $z_{\mathrm{L(R)}}$ because the variation of $z_{\mathrm{L(R)}}$ corresponds to the variation of the location of the curves $\eta = ay^3 + by + c$ in the η-direction and this does not change the property that there are three distinct real roots provided the variation is small enough (i.e. the width of the brane is small enough). These three curves (which are identical except for the translation in the y-direction) may be interpreted as three generations of fermions. Inspection of (16) reveals that the masses of all the generations of the fermions are the same. In order to break this degeneracy one may either explicitly break the degeneracy in an ad hoc way (for example in the way introduced in the following paragraphs) or one may introduce a direct y dependence into the metric. The second way is more promising. However, in that case to find an appropriate Lagrangian which satisfies the Einstein equations becomes a rather non-trivial matter. So at this step we assume degeneracy for the masses of the fermion generations (although this is not realistic). Although the $\phi_{1(2)}$–Ψ interactions do not discriminate between different fermions in the same family, gravity does, as we have seen in the previous paragraphs where we have discussed the localization of the gravitons. So in principle there are universality breaking effects due to gravitational interactions for different fermions. However, we assume that these effects are so small (i.e. the four dimensional brane is so narrow in the fifth dimension) that they cannot be detected at present. The relevant part of $\Psi_{\mathrm{L(R)}}$ at low energies is the portion of their curve in the η-direction which overlaps with the portion in the fifth coordinate where our four dimensional world is located. We denote the average location in the η-directions by $\eta = z$. So each fermion generation may be labeled by the roots of $\eta = ay^3 + by + c = z$; that is, y_i, $i = 1, 2, 3$, in general. In other words one may write

$$\begin{aligned} &\Psi_{\mathrm{L(R)}}(x,y) = \Psi_{\mathrm{L(R)}}(x,y_i) \quad \text{if} \quad y = y_i, \\ \text{and} \quad &\Psi_{\mathrm{L(R)}}(x,y) = 0 \quad \text{otherwise} \\ \text{and} \quad &\Psi_{\mathrm{L(R)}}(x,y_i) = \psi_{i\mathrm{L(R)}}(x), \end{aligned} \tag{21}$$

where $i = 1, 2, 3$ stands for the family index, y_i stands for the solutions of the equation $\eta = z$ (i.e. the locations of the families in the fifth dimension), and x stands for the usual four dimensional coordinates. The solutions in (19) are classical solutions corresponding to the intensity of the quantum fields. So the procedure of taking only the values of $\Psi_{\mathrm{L(R)}}$ at $\eta = z$ corresponds to neglecting all the quantum fields which are excited outside of our 3-brane. Of course different gauge bosons may penetrate into the bulk with different depths so that the average location of the fermions with respect to these interactions may change. So different gauge bosons serve to distinguish different types of fermions (e.g quarks from leptons) without directly referring to the gauge interactions.

Another interesting aspect of the above equations is that they have mass-like terms for chiral fermions while, as long as we are aware, the previous solutions are given for massless chiral fermions [9,12]. In this context one may generate fermion masses by using the method of the overlap of fermion wave functions or by embedding this model in a six or higher dimensional model. In higher dimensional models the mass-like terms, m, will contribute to the mass matrix which gives the masses for the physical fermions after diagonalization. In the case of six dimensions such a scheme will result in the usual fermions and their mirrors with the same masses. So physically relevant models need to assign the fermions and their mirrors to different gauge groups. In the case of seven or higher dimensional models it is possible to give the fermions and their mirrors different masses provided the entries of the mass matrix are taken as general complex numbers. To be more precise, let us consider the following seven dimensional Dirac equation:

$$\Gamma^A D_A \Psi = 0, \tag{22}$$

where

$$\begin{aligned} &\Gamma^A D_A = \begin{pmatrix} V & \mathrm{i}D_5 + D_6 \\ \mathrm{i}D_6 - D_6 & -V \end{pmatrix}, \\ &\quad V = \mathrm{i}\gamma_5 D_4 + \gamma^\mu D_\mu, \quad D_A = \partial_A + \mathrm{i}gB_A, \\ &\quad A = 0, 1, ..., 6, \quad \mu = 0, 1, 2, 3. \end{aligned}$$

If either the gauge bosons corresponding to extra dimensions have vacuum expectation values or the derivatives give mass terms due to compactification of the extra dimensions or due to both, this equation induces a mass matrix

$$M = \begin{pmatrix} \mathrm{i}\gamma_5 m_4 & \mathrm{i}m_5 + m_6 \\ \mathrm{i}m_5 - m_6 & -\mathrm{i}\gamma_5 m_4 \end{pmatrix}. \tag{23}$$

This equation leads to two different fermions, the usual fermions and their mirrors, with different masses provided one takes the m_i as arbitrary complex numbers. One may employ an orbifold symmetry [13] if one needs the fermion masses to be small. Both masses become the same if one reduces the dimension of the space-time to six or let all m_i be real.

As we have mentioned in the introduction the framework introduced here has some conceptual similarities with the study of Dvali and Shifman [3].[2] They simply assume that there is more than one brane in the fifth dimension without giving an explicit model which realizes this. They take the extra dimension to be compact and they do not consider the effect of gravity. Because of experimental constraints [15] these branes must be close in the extra dimension if they all contain the standard model particles. This makes neglecting gravity difficult and the stabilization of these branes is a more subtle question. Moreover, they give their analysis on general grounds. Of course this approach has some advantages, such as providing a general framework for future studies. However we believe that the introduction of a more specific scheme will be phenomenologically more promising. As we have mentioned in the introduction another study which takes different fermions to differ by their locations in the extra dimension(s) is given by Arkani-Hamed and Schmaltz [4]. Although they do not study the problem of the explanation of the origin of fermion families and chirality, their study has some conceptual points in common with the present one. However, they also do not consider the effect of gravity, and their space is compact. Moreover the different fermion families are simply put in different locations in the extra dimension(s) to give them different masses, while in the present model the different locations of the fermions naturally arise as a result of the non-trivial form of the domain wall. However the models by [3,4] are stronger than the present model in one respect; they obtain the fermion masses by using the technique of the overlap of wave functions as we do not introduce a method to derive the fermion masses. Moreover, this can simultaneously explain why different fermion families have different masses without going to dimensions higher than five. One can employ the same technique to obtain the fermion masses in this model. We leave this point open to facilitate consideration of different options as well in the future.

We have seen that there is considerable hope for explaining the origin of fermion families and chirality by using domain wall structures in extra dimensions. We think that one of the most important virtues of the present model is that it reaches almost all of its conclusions through explicit formulae instead of a vague picture. However there is still a long way to go to put this scheme in a more detailed phenomenological model which can give a realistic description of chirality and fermion families in the context of the standard model. Probably in such a description one should take the gauge bosons corresponding to weak interactions to be localized on the sub-brane containing the left handed brane, while the gauge bosons of

[2] See also [14]

the non-chiral interactions can freely propagate over the whole (mother) 3-brane. Such an attempt may need to embed this simple scheme in higher dimensions. Such an extension may be done by giving similar constructions and solving the corresponding equations for vortices [16] or other topological defects. Another, maybe simpler, route to go is to take the topological defect in a higher dimension to be a domain wall junction or a similar intersection of multi-branes in higher dimensions [17]. All these points will be clarified by further studies in the future.

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